Nash Equilibria in Games with Playerwise Concave Coupling Constraints: Existence and Computation

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Games with Coupling Constraints

- m players, each with convex compact strategy space $\mathcal{X}_i \subset \mathbb{R}^{d_i}$,
- joint strategy space $\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$,
- continuous utilities $u_i : \mathcal{X} \to \mathbb{R}$.
- **Shared coupling constraints:**
  - feasible strategies $\mathcal{C} := \{x \in \mathcal{X} \mid \forall j \in [b],\ c_j(x) \geq \alpha_j\}$ with
  - continuous costs $c_j : \mathcal{X} \to \mathbb{R}$, and thresholds $\alpha_j \in \mathbb{R}$.
- **Notation:** for $x_{-i} \in \mathcal{X}_{-i}$, let $\mathcal{C}_i(x_{-i}) := \{x_i \in \mathcal{X}_i \mid (x_i, x_{-i}) \in \mathcal{C}\}$.

- $x \in \mathcal{X}$ is a **constrained Nash equilibrium** if $x \in \mathcal{C}$ and $\forall i \in [m]$, $\forall x_i' \in \mathcal{C}_i(x_{-i})$, we have $u_i(x_i', x_{-i}) \leq u_i(x)$

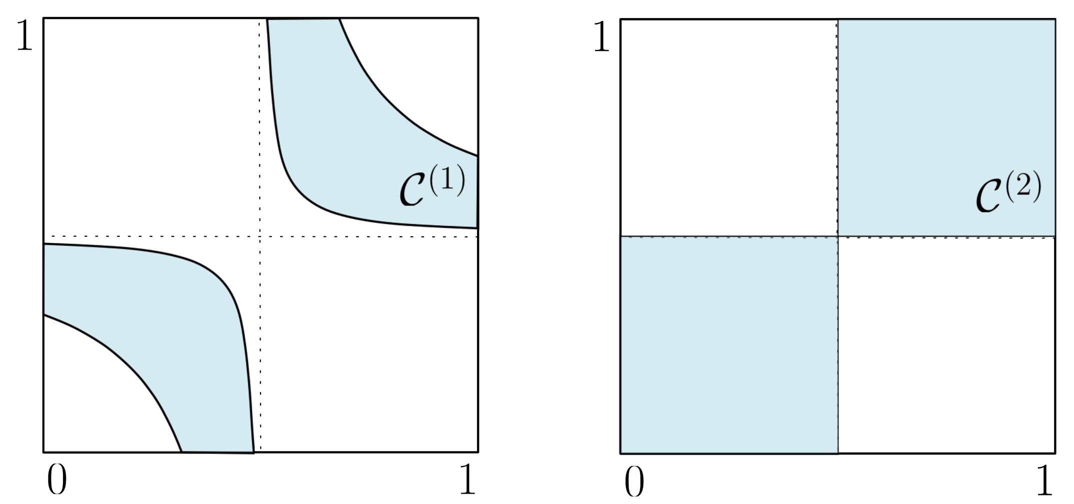
Assumption 1. $\forall i \in [m], \forall x_{-i} \in \mathcal{X}_{-i}$, the functions $x_i \mapsto u_i(x_i, x_{-i})$ and $x_i \mapsto c_j(x_i, x_{-i})$ are concave for all $j \in [b]$.

Our Contributions

- **Existence:** We prove existence of constrained Nash equilibria.
- **Computation:** We propose an independent log barrier regularized gradient method with finite time convergence to ϵ-approx. Nash equilibria for potential games with coupling constraints.

Challenges & Prior Work

Prior existence results require **(a)** feasible region $\mathcal{C}$ to be convex, or **(b)** $\mathcal{C}_i(x_{-i}) \neq \emptyset$ for any $i \in [m], x_{-i} \in \mathcal{X}_{-i}$ (neither applies to $\mathcal{C}^{(1)}$).



- For $u_1(x_1, x_2) = u_2(x_1, x_2) = x_1(1 - x_2)$ over $\mathcal{C}^{(2)}$, no constrained Nash equilibrium exists. Assumption 2 prevents gradients from vanishing on boundary: satisfied for $\mathcal{C}^{(1)}$, but not for $\mathcal{C}^{(2)}$.

Assumption 2. For any $x \in \mathcal{C}$, let $\mathcal{J}(x) := \{j \in [b] \mid c_j(x) = \alpha_j\}$. We assume $\forall x \in \mathcal{C}$ and $\forall i \in [m]$, $\exists \delta_i \in \mathbb{R}^{d_i}$ such that $\forall j \in \mathcal{J}(x)$, $\langle \delta_i, \nabla_{x_i} c_j(x) \rangle > 0$.

Existence Result

Theorem 1. Let Assumptions 1 and 2 hold and suppose $\mathcal{C} \neq \emptyset$. Then there exists a constrained Nash equilibrium $x^\star \in \mathcal{X}$.

Background.

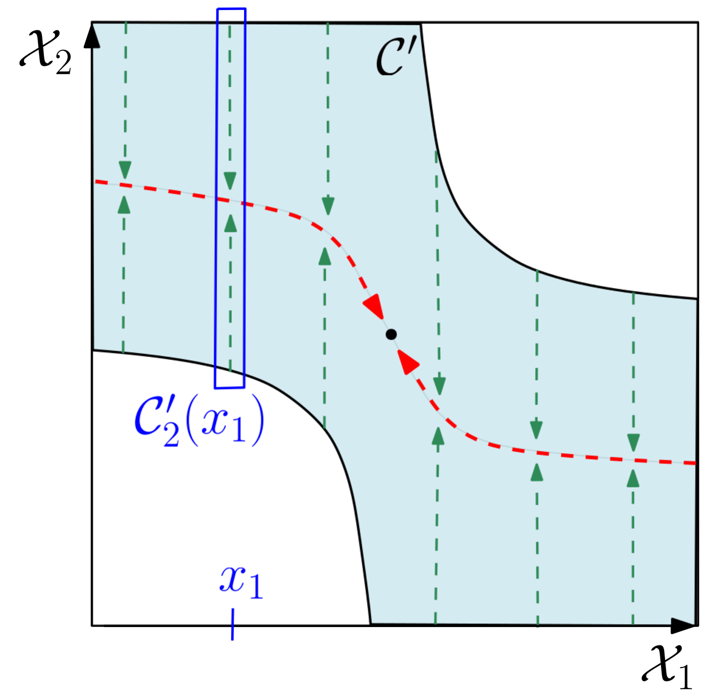
- Classical existence results apply fixed point theorems (Brouwer/Kakutani) to best response map defined over **convex** domain.
- **Fact [Begle, 1950].** If $X \subset \mathbb{R}^n$ is compact, contractible, and $\phi : X \to X$ continuous, there exists $x \in X$ such that $x = \phi(x)$.

Proof overview.

1) Show that any connected component $\mathcal{C}' \subseteq \mathcal{C}$ is contractible using playerwise concavity of constraints and Assumption 2.

2) Define continuous best response function $\Psi := \Psi_1 \circ \cdots \circ \Psi_m$ with

$$\Psi_i(x) := \left(\underset{x_i' \in \mathcal{C}_i'(x_{-i})}{\arg\max}\ u_i(x_i', x_{-i}),\ x_{-i} \right).$$



$\implies$ $x^\star \in \mathcal{C}'$ with $x^\star = \phi(x^\star)$ is a constrained Nash equilibrium.

Learning Setting

- Each round, player i chooses $x_i^{(t)} \in \mathcal{X}$ and receives feedback $\nabla_{x_i} u_i(x^{(t)})$, $c_j(x^{(t)})$, and $\nabla_{x_i} c_j(x^{(t)})$ for all $j \in [b]$.
- **Independent learning**: players cannot observe others' strategies or utility feedback; no communication/coordination allowed.

Assumptions.

- Start from initial strategy $x^{(0)}$, $c_j(x^{(0)}) \geq \alpha_j + \gamma$ for some $\gamma > 0$ (otherwise finding any feasible strategy is known to be NP-hard).
- Utilities u_i for all $i \in [m]$, and constraints c_j for all $j \in [b]$ are differentiable, M-smooth, and L-Lipschitz continuous.
- **Potential game:** there exists $\Phi : \mathcal{C} \to \mathbb{R}$ such that $\forall i \in [m], \forall x \in \mathcal{C}$, and $\forall x_i' \in \mathcal{C}_i(x_{-i})$, potential function captures utility changes,

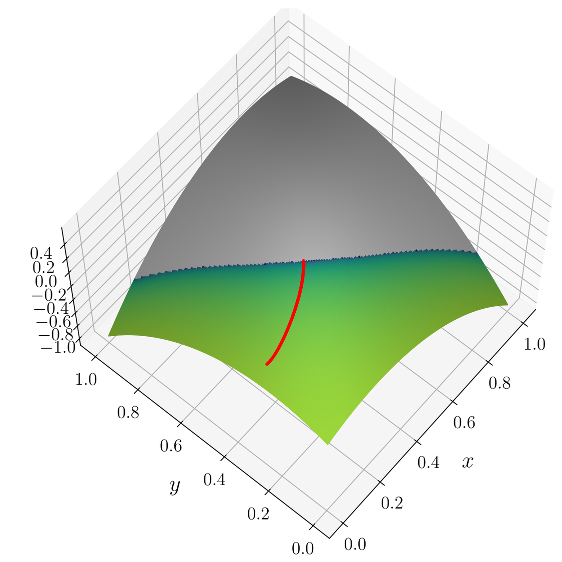
$$\Phi(x) - \Phi(x_i', x_{-i}) = u_i(x) - u_i(x_i', x_{-i}).$$

- **Extended Mangasarian-Fromovitz constraint qualification:** $\exists \rho > 0$ such that Assumption 2 holds for ρ-approximate active constraints, i.e., $\forall j \in \mathcal{J}_\rho(x) := \{j \in [b] \mid c_j(x) - \alpha_j \leq \rho\}$.

Log Barrier Method

- Define regularized objectives for $\eta > 0$, and each $i \in [m]$,

$$B_i^\eta(x) := u_i(x) + \eta \sum_{j \in [b]} \log(c_j(x) - \alpha_j).$$



- For $t = 0, \ldots, T-1$ each player $i \in [m]$ simultaneously updates

$$x_i^{(t+1)} = \mathcal{P}_{\mathcal{X}_i}\left[x^{(t)} + \gamma^{(t)} \nabla_{x_i} B_i^\eta(x)\right]. \qquad (1)$$

- **Challenge:** $B_i^\eta(x)$ is **not** globally smooth.
  - **Solution:** for adaptively chosen $\gamma_i^{(t)}$, we can show smoothness along strategy trajectory $x^{(t)}, 0 \leq t \leq T$.

Convergence Result

Theorem 2. Let $\epsilon > 0$ and set $\eta = \epsilon$. Under the stated assumptions and appropriate choices of $\{\gamma^{(t)}\}_{t=0}^{T-1}$, suppose we perform $T = \mathcal{O}(\epsilon^{-3})$ rounds of updates as in (1). Then, there exists $0 \leq t \leq T$ such that $x^{(t)}$ is a constrained ϵ-approximate Nash equilibrium.

Proof overview.

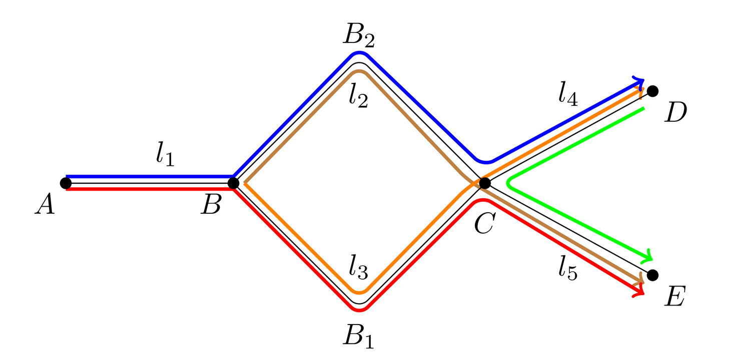
- $\Phi^\eta(x) := \Phi(x) + \eta \sum_{j \in [b]} \log(c_j(x) - \alpha_j)$ is a potential function for the regularized game with utilities $B_i^\eta(x)$.
- Local bound on $\|\nabla^2 \Phi^\eta(x)\| \implies$ smoothness along trajectory.
- Convergence to ϵ-stationary point $x^\star$ of Φ^η.
  - $\implies$ ϵ-approximate KKT conditions for $\max_{x \in \mathcal{C}'} \Phi^\eta(x)$ at $x^\star$.
  - $\implies$ $x^\star$ is constrained ϵ-Nash equilibrium.

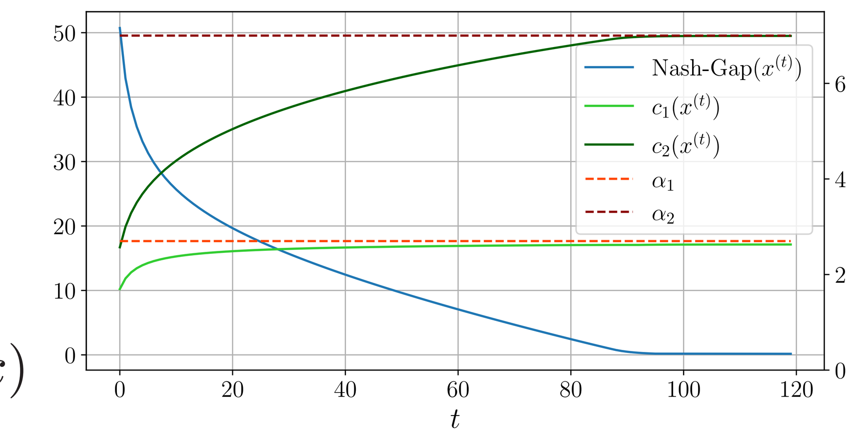
Simulation

Network routing game

- players aiming to route flow along colored paths in network; utilities depend on link congestion
- coupled link capacity constraints (here on l_2 and l_5)
- observe converge of

$$\text{Nash-Gap}(x) := \max_{i \in [n]} \max_{x_i' \in \mathcal{C}_i(x_{-i})} u_i(x_i', x_{-i}) - u_i(x)$$





[Begle, 1950] Edward G Begle. A fixed point theorem. *Annals of Mathematics*, 51(3):544–550, 1950.